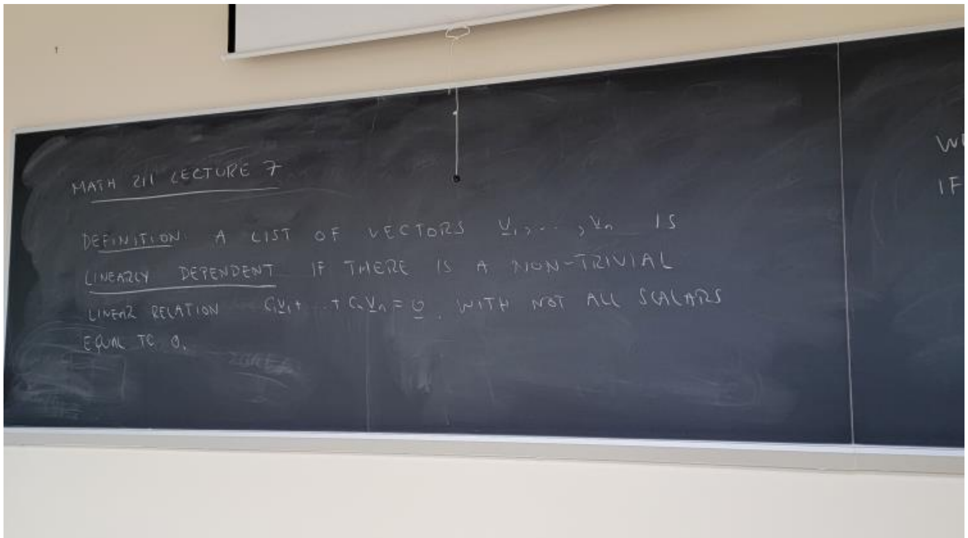
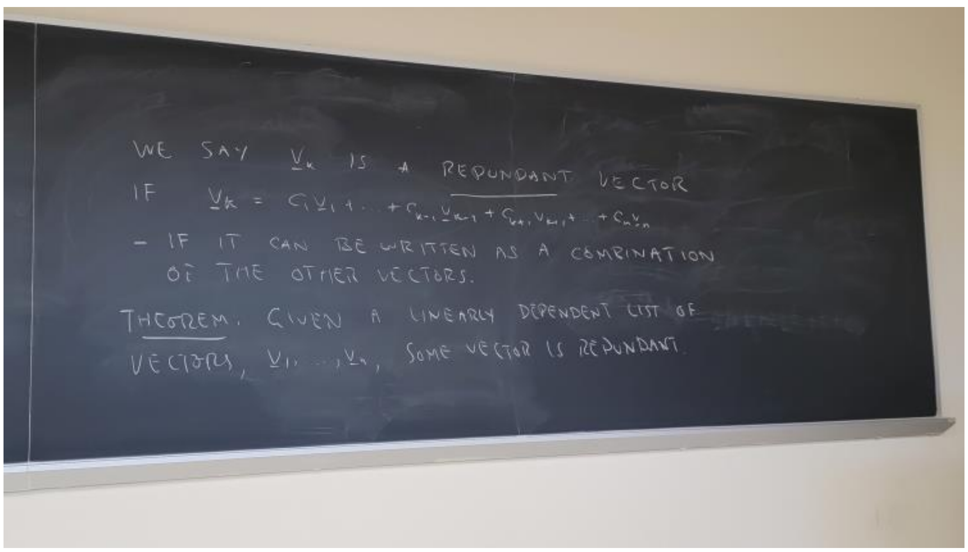
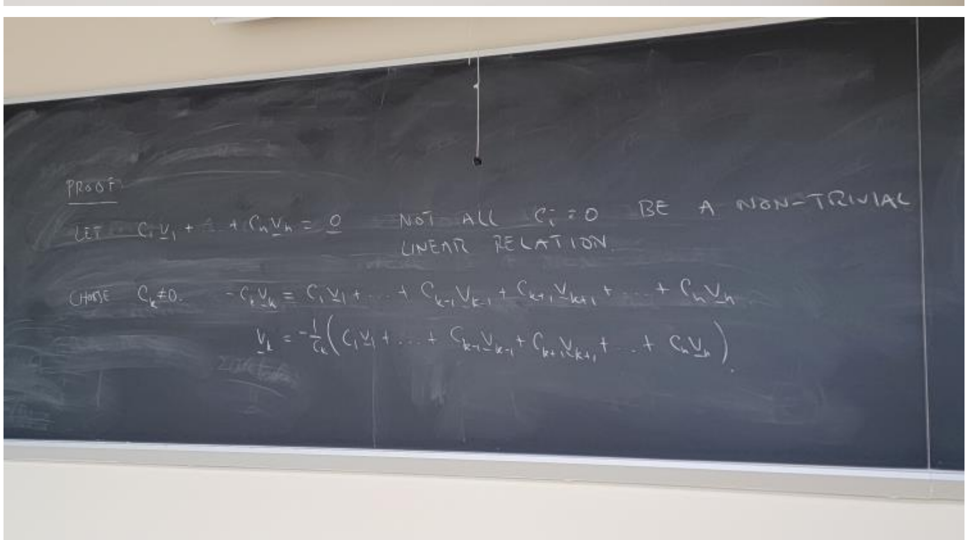
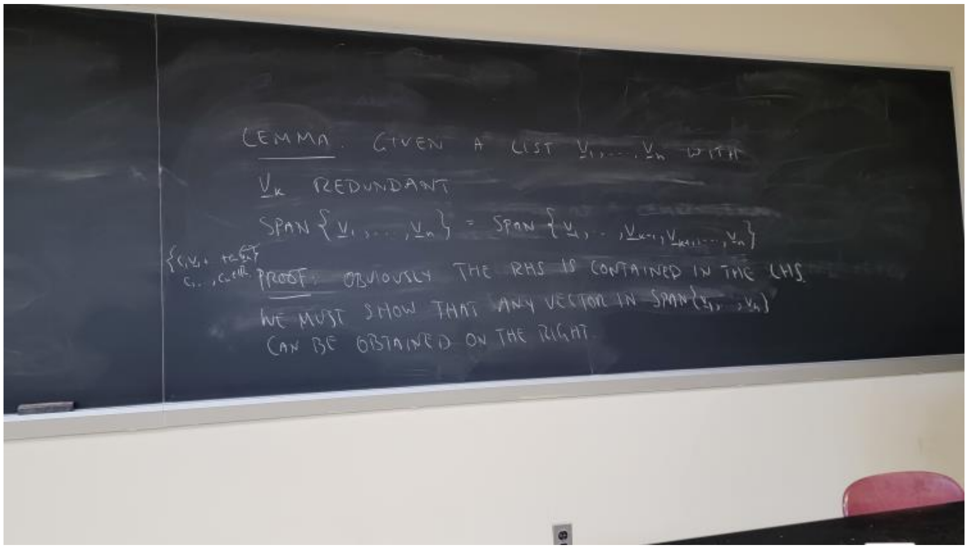
Lecture #6 (2/14/23)

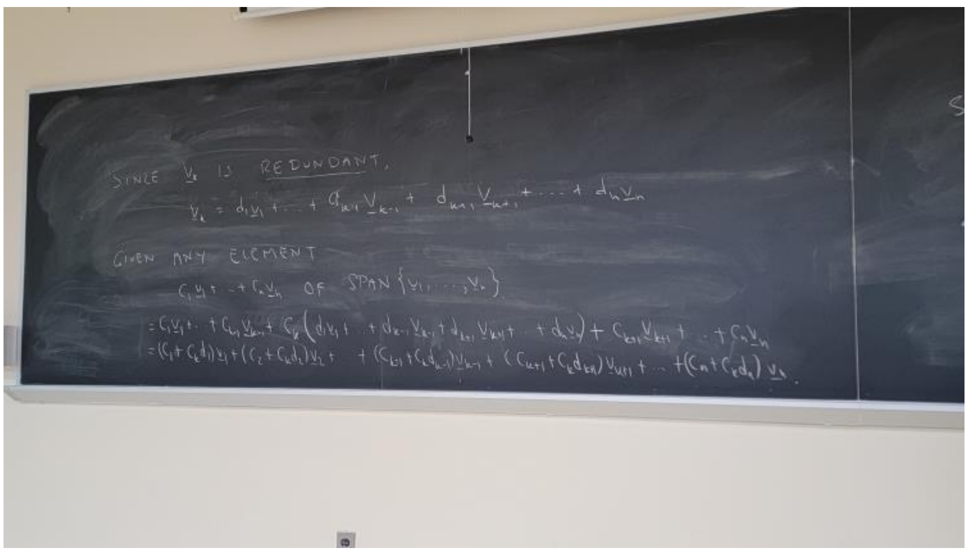
Tuesday, February 14, 2023 2:54 PM

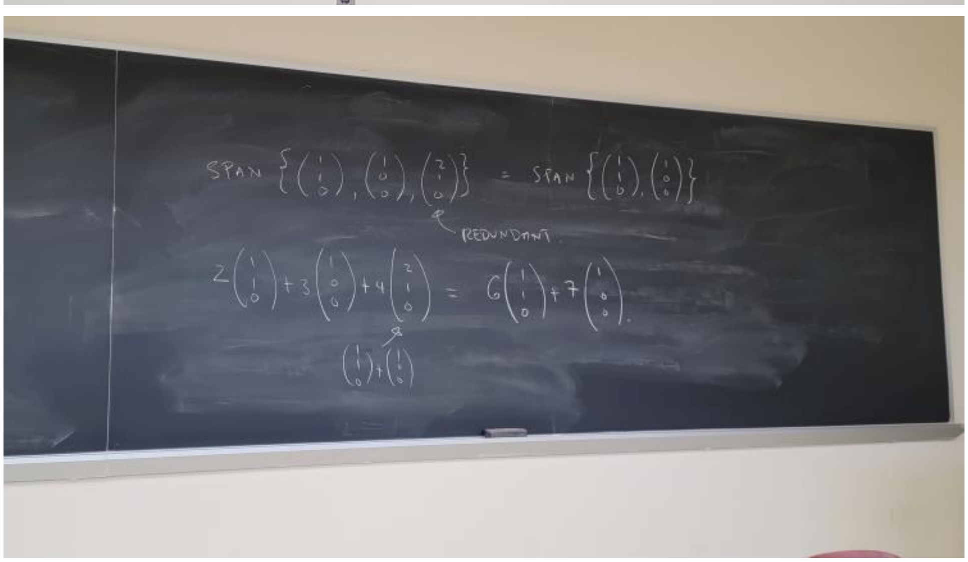


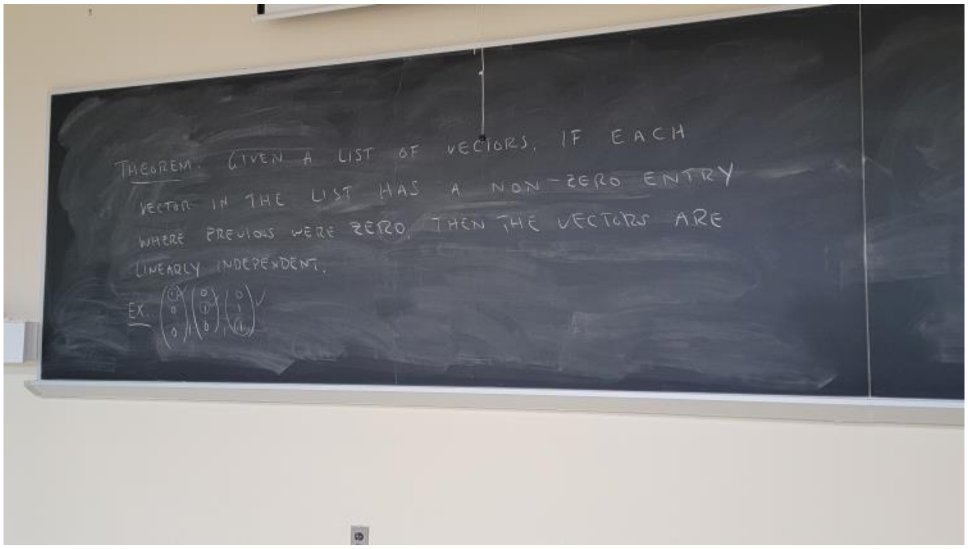


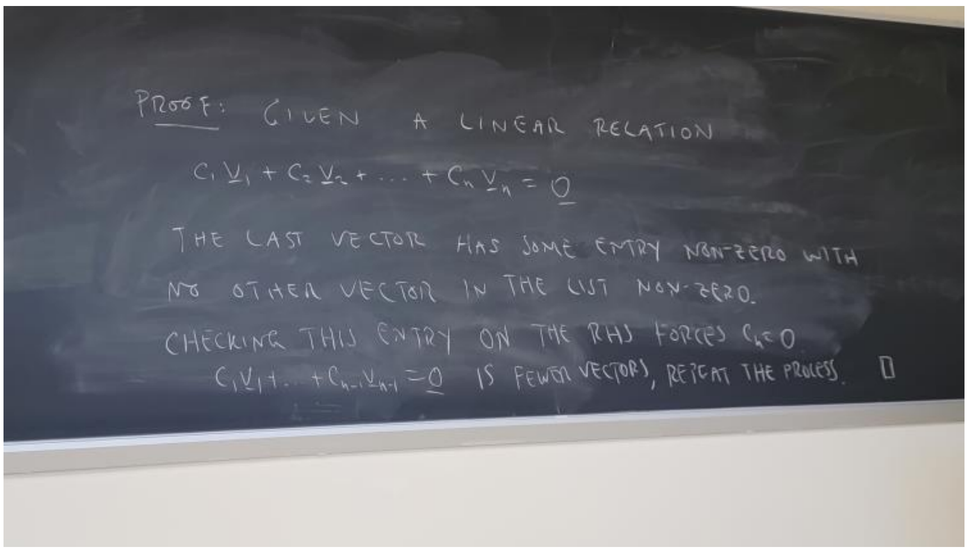


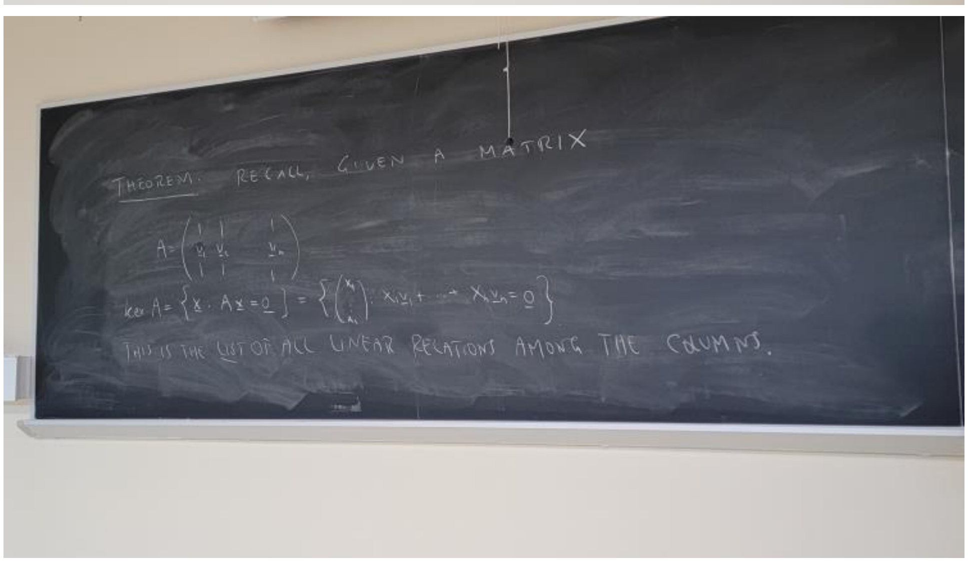

LEMMA. GIVEN A LIST v1, ..., vn WITH
vk REDUNDANT
SPAN {v1, ..., vn} = SPAN {v1, ..., vk-1, vk+1, ..., vn}
{c1v1 + ... + cnvn | c1, ..., cn ∈ ℝ}
PROOF: OBVIOUSLY THE RHS IS CONTAINED IN THE LHS
WE MUST SHOW THAT ANY VECTOR IN SPAN{v1, ..., vn}
CAN BE OBTAINED ON THE RIGHT


SINCE vk IS REDUNDANT,
vk = d1v1 + ... + dk-1vk-1 + dk+1vk+1 + ... + dnvn
GIVEN ANY ELEMENT
c1v1 + ... + cnvn OF SPAN {v1, ..., vn}
= c1v1 + ... + ck-1vk-1 + ck(d1v1 + ... + dk-1vk-1 + dk+1vk+1 + ... + dnvn) + ck+1vk+1 + ... + cnvn
= (c1 + ckd1)v1 + (c2 + ckd2)v2 + ... + (ck-1 + ckdk-1)vk-1 + (ck+1 + ckdk+1)vk+1 + ... + (cn + ckdn)vn.


SPAN {(1,1,0), (1,0,0), (2,1,0)} = SPAN {(1,1,0), (1,0,0)}
REDUNDANT
2(1,1,0) + 3(1,0,0) + 4(2,1,0) = 6(1,1,0) + 7(1,0,0).
(1,1,0) + (1,0,0)


THEOREM: GIVEN A LIST OF VECTORS, IF EACH
VECTOR IN THE LIST HAS A NON-ZERO ENTRY
WHERE PREVIOUS WERE ZERO, THEN THE VECTORS ARE
LINEARLY INDEPENDENT.
EX.


PROOF: GIVEN A LINEAR RELATION
$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_n \underline{v}_n = \underline{0}$
THE LAST VECTOR HAS SOME ENTRY NON-ZERO WITH
NO OTHER VECTOR IN THE LIST NON-ZERO.
CHECKING THIS ENTRY ON THE RHS FORCES $c_n = 0$
$c_1 \underline{v}_1 + \dots + c_{n-1} \underline{v}_{n-1} = \underline{0}$ IS FEWER VECTORS, REPEAT THE PROCESS. □


THEOREM: RECALL, GIVEN A MATRIX
$A = \begin{pmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{pmatrix}$
$\ker A = \{ \underline{x} : A\underline{x} = \underline{0} \} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_1 \underline{v}_1 + \dots + x_n \underline{v}_n = \underline{0} \right\}$
THIS IS THE LIST OF ALL LINEAR RELATIONS AMONG THE COLUMNS.

DEFINITION. IF V IS A SUBSPACE OF $\mathbb{R}^n$, $\underline{v}_1, \ldots, \underline{v}_k$ ARE A BASIS FOR V IF SPAN $\{\underline{v}_1, \ldots, \underline{v}_k\} = V$ AND $\underline{v}_1, \ldots, \underline{v}_k$ ARE LINEARLY INDEPENDENT.

THEOREM $\{\underline{v}_1, \ldots, \underline{v}_k\}$ ARE A BASIS FOR A SUBSPACE V OF $\mathbb{R}^n$ IF AND ONLY IF EACH $\underline{v} \in V$ HAS A UNIQUE REPRESENTATION $\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_k\underline{v}_k$ AND ALSO EACH $\underline{v}_1, \ldots, \underline{v}_k \in V$

PROOF: SUPPOSE $\underline{v}_1, \ldots, \underline{v}_k$ ARE A BASIS.

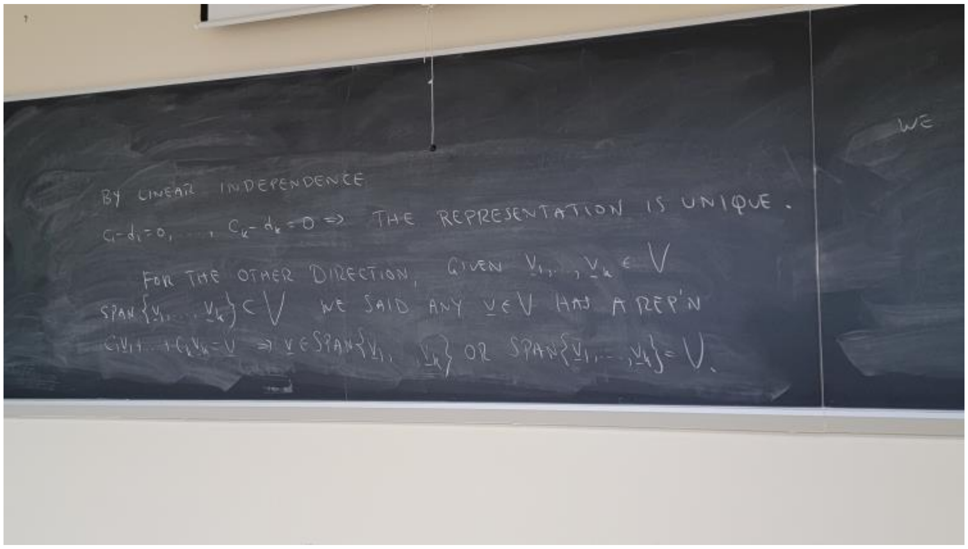
SPAN $\{\underline{v}_1, \ldots, \underline{v}_k\} = V$.

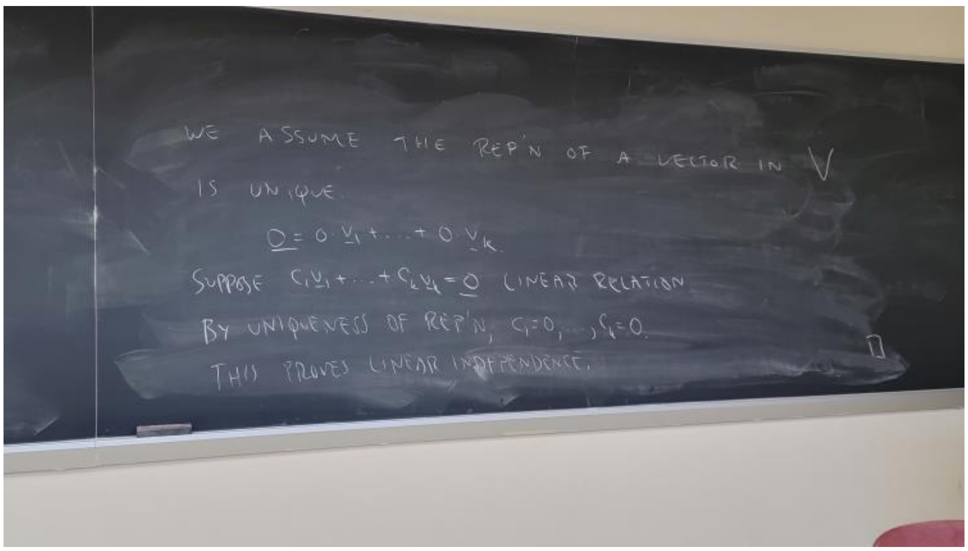
GIVEN $\underline{v} \in V$, THERE EXIST SCALARS $c_1, \ldots, c_k$

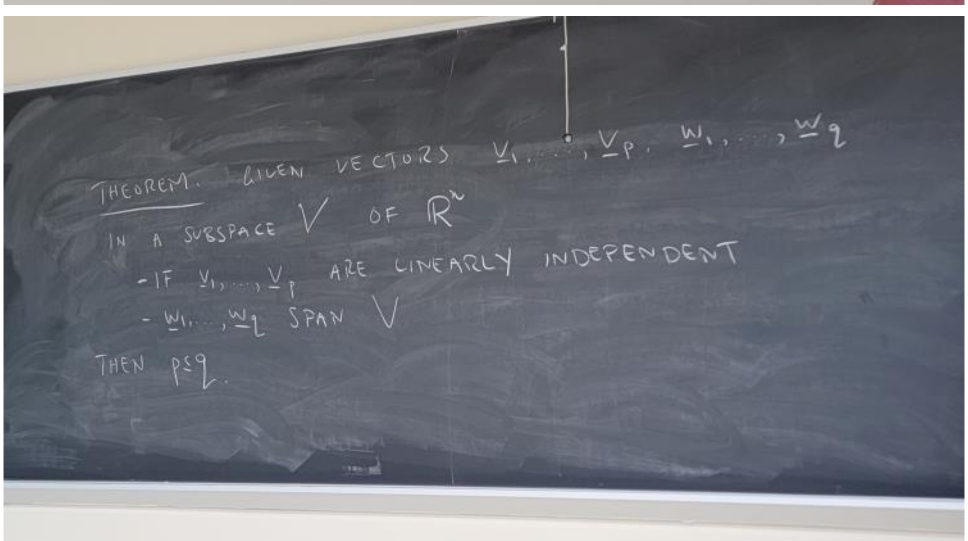
$c_1\underline{v}_1 + \cdots + c_k\underline{v}_k = \underline{v}$

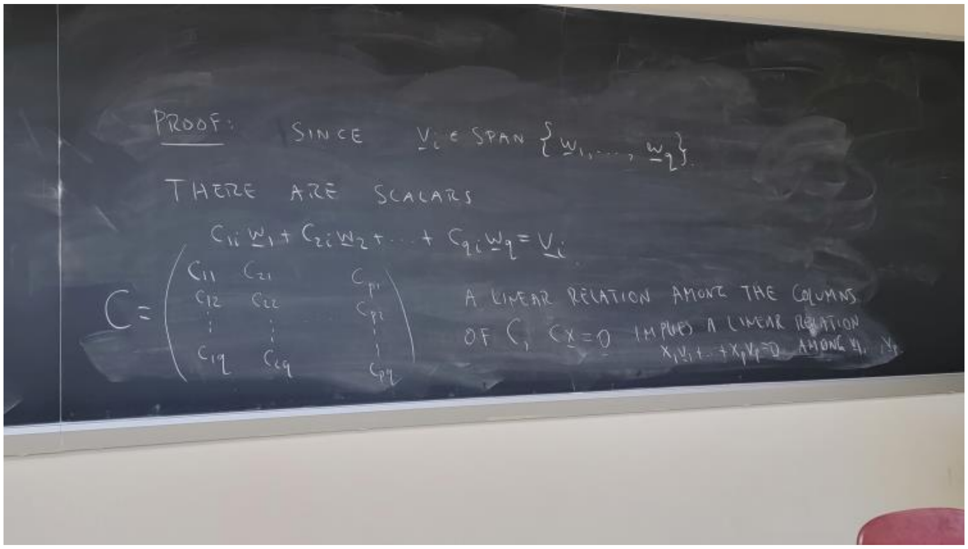
SUPPOSE $d_1\underline{v}_1 + \cdots + d_k\underline{v}_k = \underline{v}$ IS ANOTHER REPRESENTATION

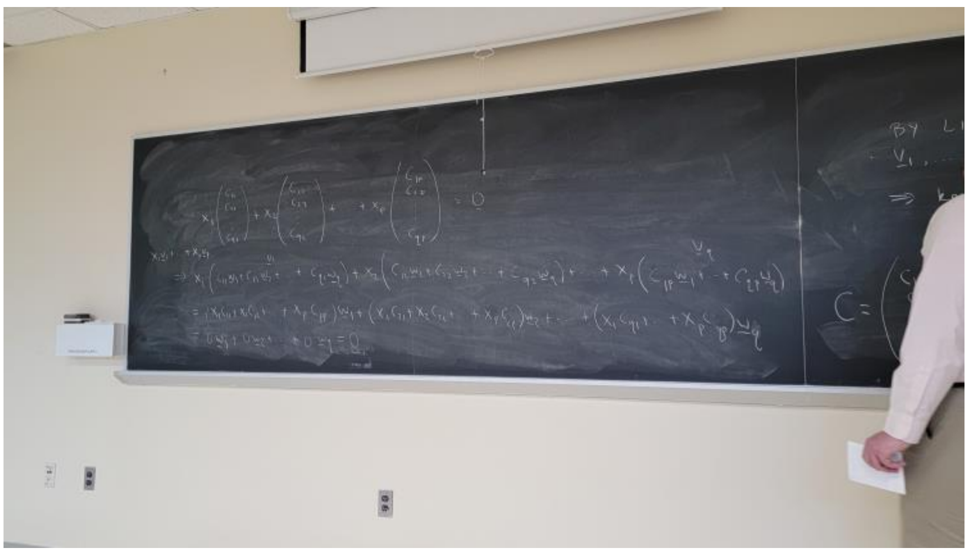
SUBTRACTING $(c_1 - d_1)\underline{v}_1 + \cdots + (c_k - d_k)\underline{v}_k = \underline{0}$, LINEAR RELATION.

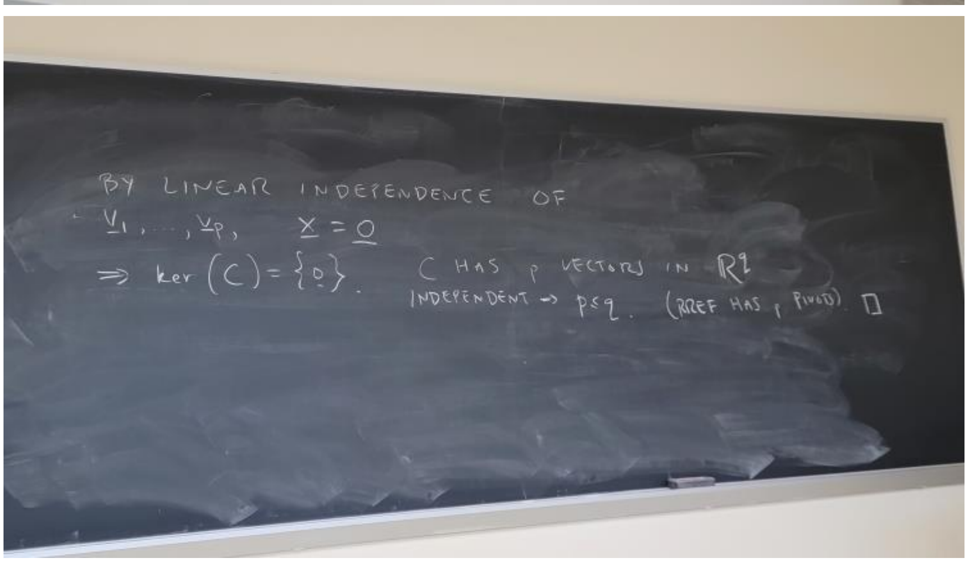

WE
BY LINEAR INDEPENDENCE
$c_1-d_1=0, \ldots, c_k-d_k=0 \Rightarrow$ THE REPRESENTATION IS UNIQUE.
FOR THE OTHER DIRECTION, GIVEN $\underline{v}_1, \ldots, \underline{v}_k \in V$
$\text{SPAN}\{\underline{v}_1, \ldots, \underline{v}_k\} \subset V$ WE SAID ANY $\underline{v} \in V$ HAS A REP'N
$c_1\underline{v}_1 + \ldots + c_k\underline{v}_k = \underline{v} \Rightarrow \underline{v} \in \text{SPAN}\{\underline{v}_1, \ldots, \underline{v}_k\}$ OR $\text{SPAN}\{\underline{v}_1, \ldots, \underline{v}_k\} = V$.

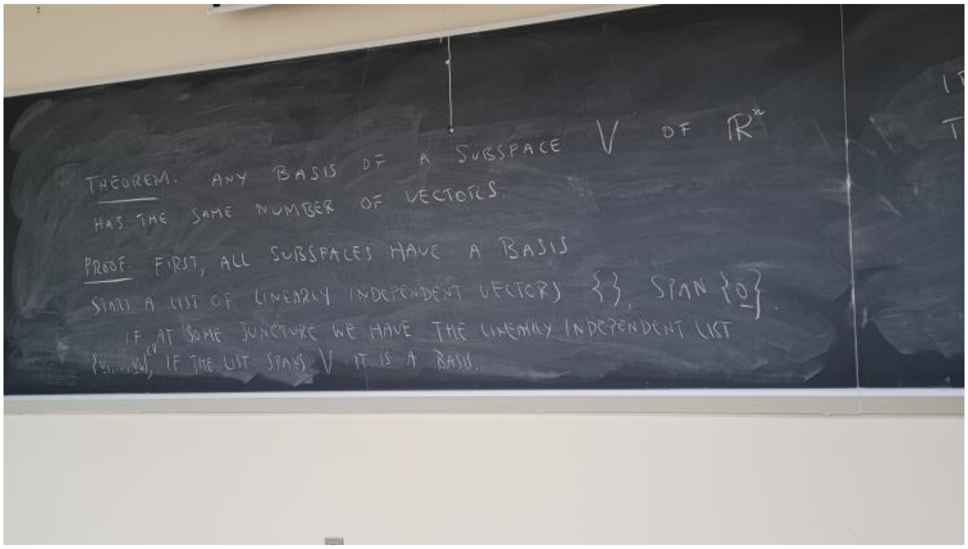

WE ASSUME THE REP'N OF A VECTOR IN V
IS UNIQUE.
$\underline{0} = 0 \cdot \underline{v}_1 + \ldots + 0 \cdot \underline{v}_k$.
SUPPOSE $c_1\underline{v}_1 + \ldots + c_k\underline{v}_k = \underline{0}$ LINEAR RELATION
BY UNIQUENESS OF REP'N, $c_1 = 0, \ldots, c_k = 0$.
THIS PROVES LINEAR INDEPENDENCE.
$\square$

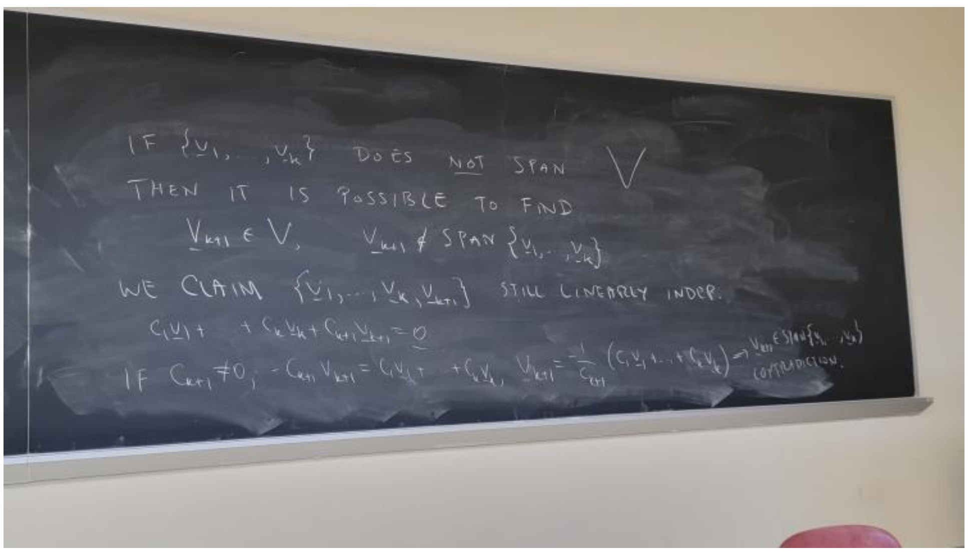

THEOREM. GIVEN VECTORS $\underline{v}_1, \ldots, \underline{v}_p$, $\underline{w}_1, \ldots, \underline{w}_q$
IN A SUBSPACE V OF $\mathbb{R}^n$
- IF $\underline{v}_1, \ldots, \underline{v}_p$ ARE LINEARLY INDEPENDENT
- $\underline{w}_1, \ldots, \underline{w}_q$ SPAN V
THEN $p \le q$.

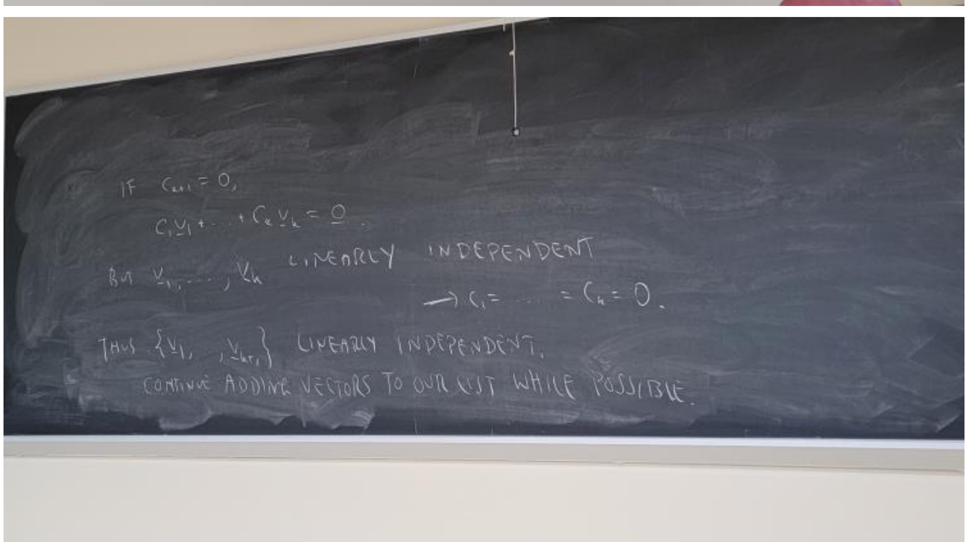

PROOF: SINCE $\underline{v}_i \in \text{SPAN}\{\underline{w}_1, \dots, \underline{w}_q\}$
THERE ARE SCALARS
$C_{1i}\underline{w}_1 + C_{2i}\underline{w}_2 + \dots + C_{qi}\underline{w}_q = \underline{v}_i$
$C = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{p1} \\ C_{12} & C_{22} & \cdots & C_{p2} \\ \vdots & \vdots & & \vdots \\ C_{1q} & C_{2q} & \cdots & C_{pq} \end{pmatrix}$
A LINEAR RELATION AMONG THE COLUMNS OF C, $C\underline{x} = \underline{0}$ IMPLIES A LINEAR RELATION
$x_1\underline{v}_1 + \dots + x_p\underline{v}_p = \underline{0}$ AMONG $\underline{v}_1, \dots, \underline{v}_p$

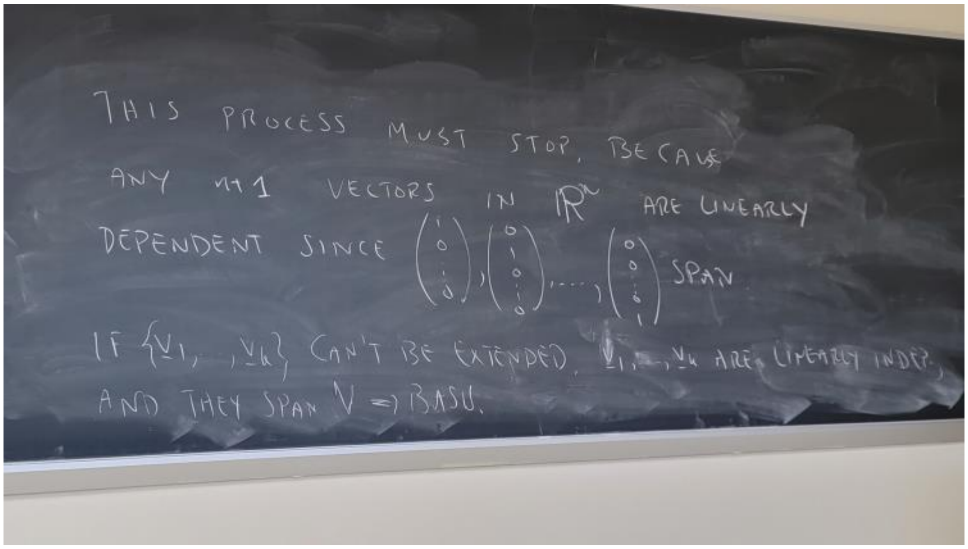
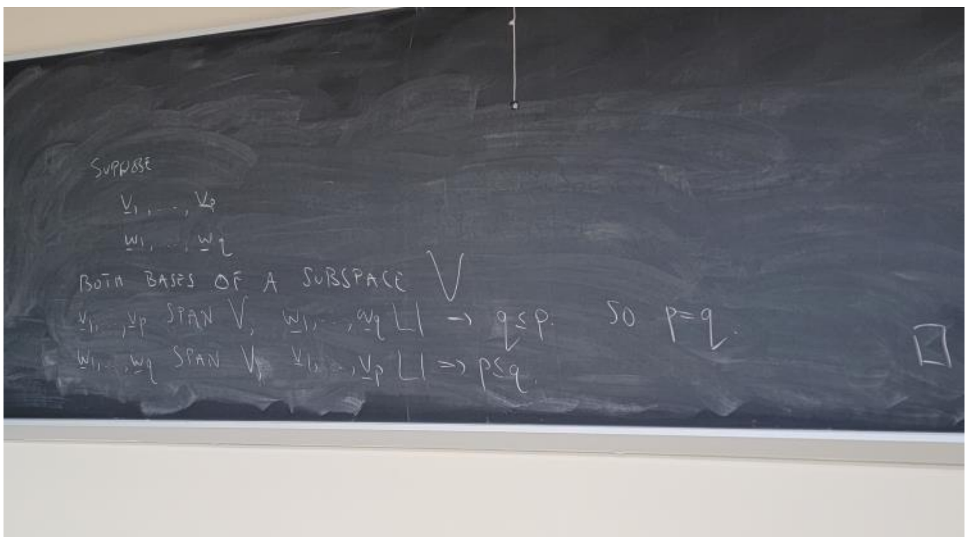
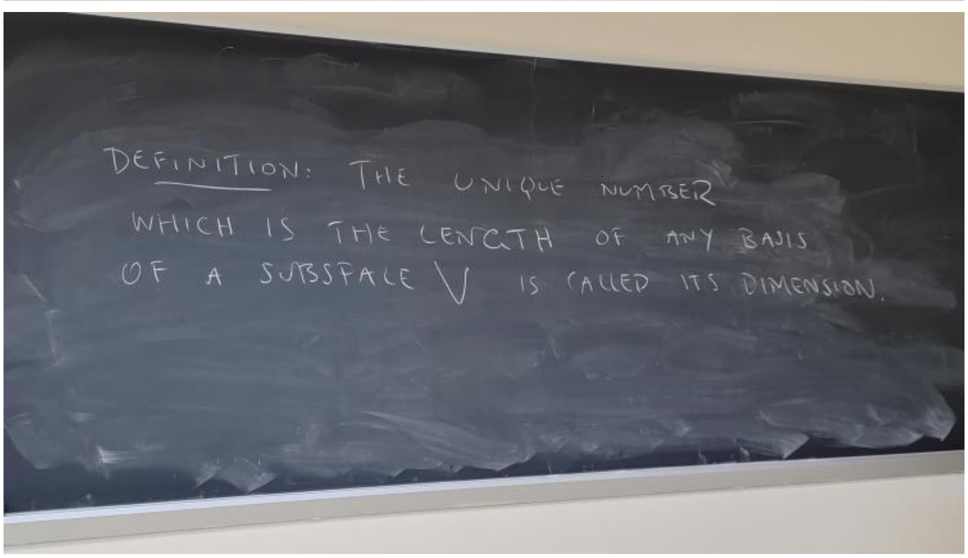


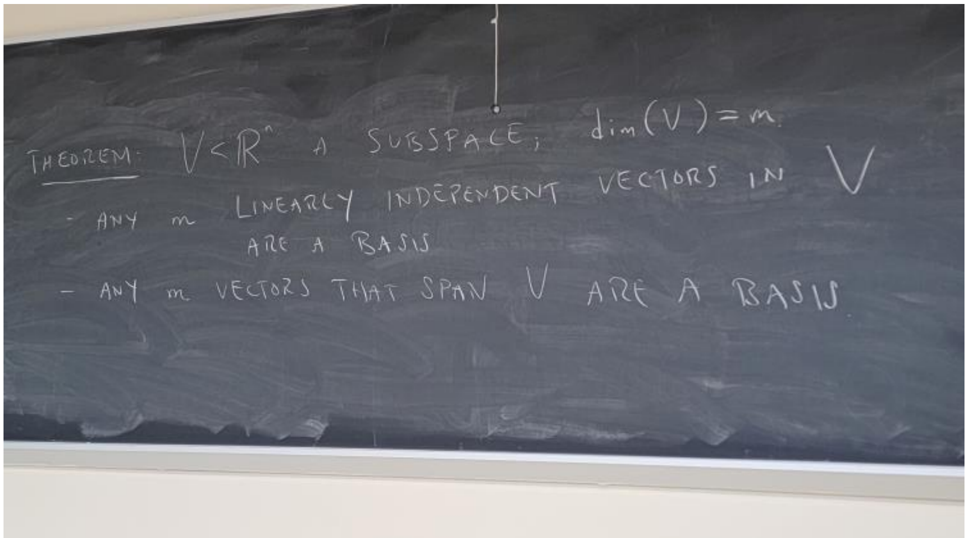

BY LINEAR INDEPENDENCE OF
$\underline{v}_1, \dots, \underline{v}_p$, $\underline{x} = \underline{0}$
$\Rightarrow \ker(C) = \{\underline{0}\}$. C HAS p VECTORS IN $\mathbb{R}^q$
INDEPENDENT $\Rightarrow p \le q$. (RREF HAS p PIVOTS) □

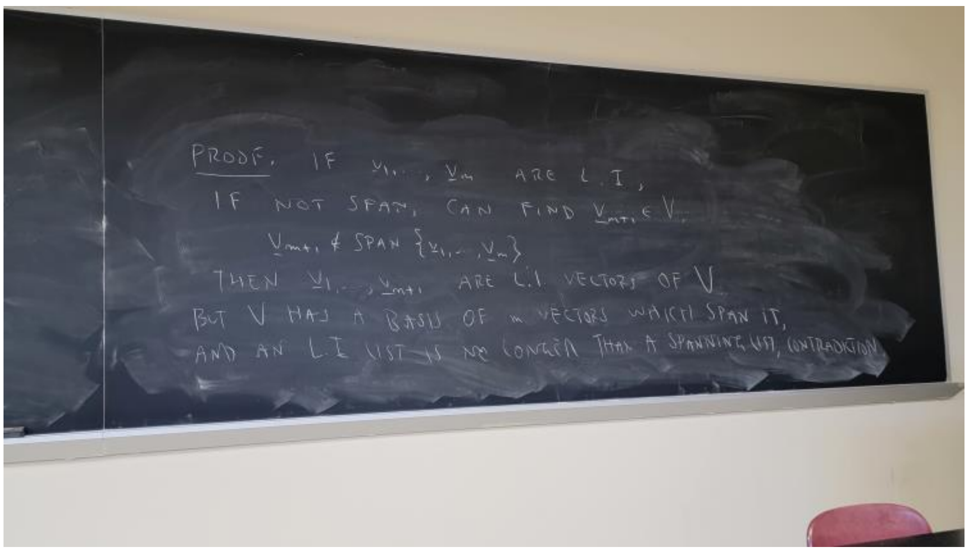

THEOREM. ANY BASIS OF A SUBSPACE V OF R^n
HAS THE SAME NUMBER OF VECTORS.
PROOF. FIRST, ALL SUBSPACES HAVE A BASIS
START A LIST OF LINEARLY INDEPENDENT VECTORS {}, SPAN {0}.
IF AT SOME JUNCTURE WE HAVE THE LINEARLY INDEPENDENT LIST
IF THE LIST SPANS V IT IS A BASIS.

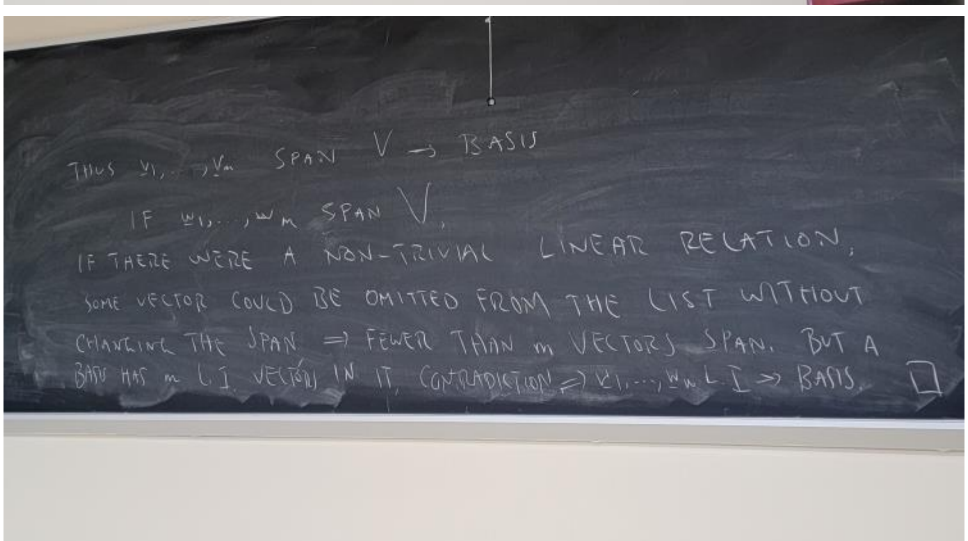

IF {v1, ..., vk} DOES NOT SPAN V
THEN IT IS POSSIBLE TO FIND
vk+1 ∈ V, vk+1 ∉ SPAN {v1, ..., vk}
WE CLAIM {v1, ..., vk, vk+1} STILL LINEARLY INDEP.
c1v1 + ... + ckvk + ck+1vk+1 = 0
IF ck+1 ≠ 0, -ck+1vk+1 = c1v1 + ... + ckvk, vk+1 = -1/ck+1 (c1v1 + ... + ckvk) → vk+1 ∈ SPAN{v1, ..., vk}
CONTRADICTION.

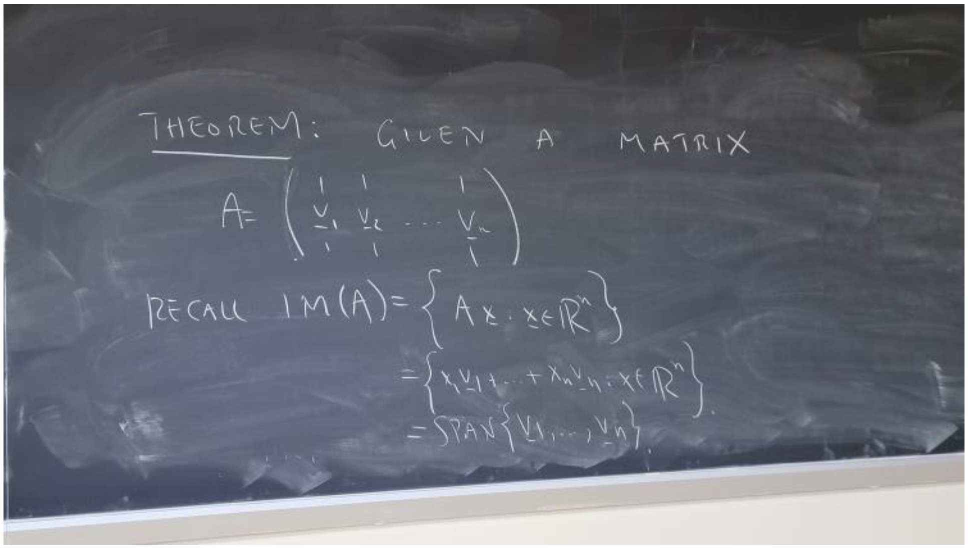

IF ck+1 = 0,
c1v1 + ... + ckvk = 0.
BUT v1, ..., vk LINEARLY INDEPENDENT
→ c1 = ... = ck = 0.
THUS {v1, ..., vk+1} LINEARLY INDEPENDENT.
CONTINUE ADDING VECTORS TO OUR LIST WHILE POSSIBLE.

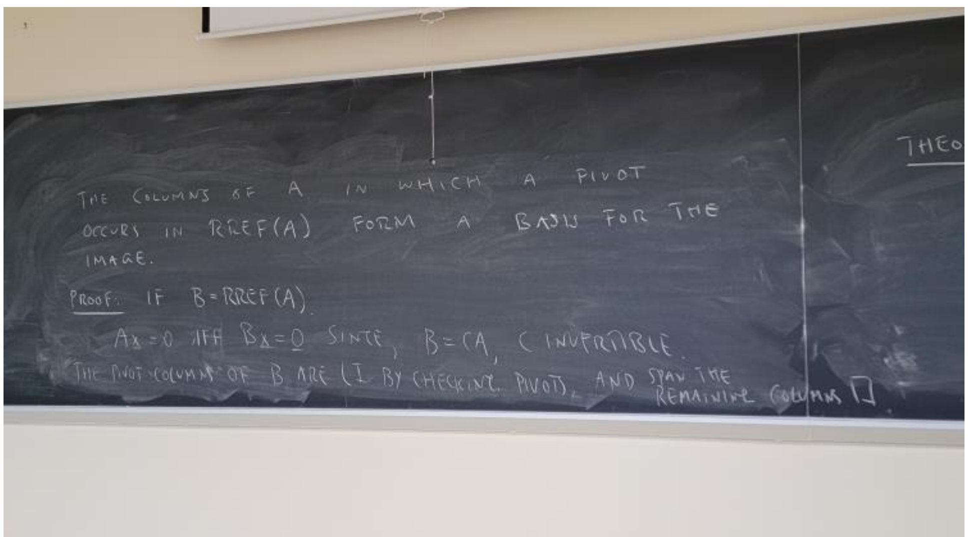

THIS PROCESS MUST STOP, BECAUSE
ANY n+1 VECTORS IN $\mathbb{R}^n$ ARE LINEARLY
DEPENDENT SINCE $\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\ \vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\0\\ \vdots\\1\end{pmatrix}$ SPAN.
IF $\{\underline{v}_1, \dots, \underline{v}_k\}$ CAN'T BE EXTENDED, $\underline{v}_1, \dots, \underline{v}_k$ ARE LINEARLY INDEP.,
AND THEY SPAN $V \Rightarrow$ BASIS.

SUPPOSE
$\underline{v}_1, \dots, \underline{v}_p$
$\underline{w}_1, \dots, \underline{w}_q$
BOTH BASES OF A SUBSPACE V
$\underline{v}_1, \dots, \underline{v}_p$ SPAN V, $\underline{w}_1, \dots, \underline{w}_q$ LI $\to q \leq p$. SO $p = q$.
$\underline{w}_1, \dots, \underline{w}_q$ SPAN V, $\underline{v}_1, \dots, \underline{v}_p$ LI $\Rightarrow p \leq q$.

DEFINITION: THE UNIQUE NUMBER
WHICH IS THE LENGTH OF ANY BASIS
OF A SUBSPACE V IS CALLED ITS DIMENSION.


THEOREM: V<R^n A SUBSPACE; dim(V) = m
- ANY m LINEARLY INDEPENDENT VECTORS IN V ARE A BASIS
- ANY m VECTORS THAT SPAN V ARE A BASIS.


PROOF. IF v1, ..., vm ARE L.I,
IF NOT SPAN, CAN FIND vm+1 ∈ V,
vm+1 ∉ SPAN {v1, ..., vm}
THEN v1, ..., vm+1 ARE L.I VECTORS OF V.
BUT V HAS A BASIS OF m VECTORS WHICH SPAN IT,
AND AN L.I LIST IS NO LONGER THAN A SPANNING LIST, CONTRADICTION


THUS v1, ..., vm SPAN V → BASIS
IF w1, ..., wm SPAN V,
IF THERE WERE A NON-TRIVIAL LINEAR RELATION,
SOME VECTOR COULD BE OMITTED FROM THE LIST WITHOUT
CHANGING THE SPAN ⇒ FEWER THAN m VECTORS SPAN. BUT A
BASIS HAS m L.I. VECTORS IN IT, CONTRADICTION ⇒ w1, ..., wm L.I ⇒ BASIS. □


THEOREM: GIVEN A MATRIX
$A = \begin{pmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{pmatrix}$
RECALL $IM(A) = \{ A\underline{x} : \underline{x} \in \mathbb{R}^n \}$
$= \{ x_1\underline{v}_1 + \cdots + x_n\underline{v}_n : x \in \mathbb{R}^n \}$.
$= SPAN\{\underline{v}_1, \ldots, \underline{v}_n\}$


THE COLUMNS OF A IN WHICH A PIVOT OCCURS IN RREF(A) FORM A BASIS FOR THE IMAGE.
PROOF: IF B = RREF(A).
$A\underline{x} = 0$ IFF $B\underline{x} = \underline{0}$ SINCE, $B = CA$, C INVERTIBLE.
THE PIVOT COLUMNS OF B ARE LI BY CHECKING PIVOTS, AND SPAN THE REMAINING COLUMNS □
THEO